

The Quantum State Of The Universe From Deformation Quantization and Classical-Quantum Correlation

M. Rashki¹ and S. jalalzadeh^{1,2*}

Department Of Physics, Shahid Beheshti University, G. C., Evin, Tehran, 19839, Iran.

Federal University of Latin-American Integration,

Technological Park of Itaipu PO box 2123, Foz do Iguaçu-PR, 85867-670, Brazil.

December 12, 2016

Abstract

In this paper we study the quantum cosmology of homogeneous and isotropic cosmology, via the Weyl-Wigner-Groenewold-Moyal formalism of phase space quantization, with perfect fluid as a matter source. The corresponding quantum cosmology is described by the Moyal-Wheeler-DeWitt equation which has exact solutions in Moyal phase space, resulting in Wigner quasiprobability distribution functions peaking around the classical paths for large values of scale factor. We show that the Wigner functions of these models are peaked around the non-singular universes with quantum modified density parameter of radiation.

Keywords: Quantum cosmology, Moyal-Wheeler-DeWitt equation, Deformation quantization

1 Introduction

The main prediction of the quantum state of the universe is the emergence of classical universe that is a manifest fact of the observable Universe. Therefore, predicting classical cosmology is a constraint on the theory of the state [1]. In minisuperspace models of cosmology, the classical gravitational field equations are often non-linear and the quantization procedure is not unique. There are infinite number of transformations that recast such equations into forms with different finite number of degree of freedom interpretations. Distinctly an appropriate transformation, state and quantization should be those that have some chance of yielding a classical limit not too far removed the original classical predictions [2].

By canonical quantization of gravity, one finds various defects when one inspects its content in detail. First, since the canonical Hamiltonian of gravity is written as a linear combination of the constraints, it annihilates the physical quantum state. Hence, the time evolution is lost from theory. Besides a new problems arises when one tries to apply the quantum theory of gravity to cosmology: when our working tool is the wave function, which should be obtained by Wheeler-DeWitt (WDW) equation or path integral, we need to know how it is possible to construct an adequate wave packet that would peaked around the original classical cosmological model [3]. In ordinary circumstances, the wave packet reduction in the Copenhagen interpretation gives rise to no practical problem as far as we regard the quantum mechanics as describing the dynamics of an ensemble of identical systems. However, in cosmology, there is only one Universe as a system. Therefore, it is not clear how the state of the Universe has any well-defined wave function. Also, in quantum cosmology the observer itself is an element of Universe. In the standard interpretation of quantum mechanics, during an observation, the quantum system must interact with a classical domain (measuring instrument, observer body, ...). In von Neumann's view, the necessity of a classical domain comes from the way it solves the measurement problem [4]. In a conventional impulsive measurement, where the coupling interaction between the measured system and the classical measuring device is of short duration and strong, the wave function plus measuring device splits into many branches which do not overlap, each one containing the measured system in an eigenstate of the measured observable, and the "pointer" of the measuring device pointing to the corresponding eigenvalue. However, in the end of the measurement processes, the observer measures only one eigenvalue and the immediate repeating of measurement gives the same result. Therefore, the wave function collapses into an eigenstate of the observable that is registered and the other branches disappear. But, a real collapse of wave function cannot be described by the unitary quantum evolution. Therefore, the Copenhagen interpretation needs to assume the existence of fundamental process in a measurement which occur outside the quantum

*email: shahram.jalalzadeh@unila.edu.br

system, in a classical domain. It is obvious that in quantum cosmology, as a quantum theory of whole Universe, there is no place for classical domain outside of it. Hence, it seems that the Copenhagen interpretation cannot be applied to quantum cosmology or it should be improved by means of further concepts. One possibility is invoking environmental decoherence [5]. However, quantum decoherence is not yet a complete answer to the measurement problem [6]. It does not explain the apparent collapse after the measurement is completed, or why all but one of the diagonal elements of the density matrix become null when the measurement is finished [7]. Also, in its developments like the consistency histories approach [8] the important role played by observers is not yet explained [9] and it is not clear how to describe a quantum universe when the background geometry is not classical [10]. Nevertheless, there are some alternative solutions to the above quantum cosmological difficulties, which together with decoherence may solve the measurement problem maintaining the universality of quantum theory and emergence of classical universe. In this line we should cite the de Broglie-Bohm interpretation of quantum cosmology [11], quantum Hamilton-Jacobi cosmology [12] and deformation quantization of cosmology [13].

The advantage of deformation quantization is that it makes quantum cosmology look like the Hamiltonian formalism of cosmology. In other words, deformation quantization can be viewed as a deformation of the structure of the algebra of classical observables, rather than a radical change in the nature of the observables. This is done by avoiding the operator formalism [13]. Deformation quantization, which is presented as Weyl-Wigner-Groenewold-Moyal phase space quantization, describes a quantum system in terms of the classical c -number variables [14, 15]. This means that operators are mapped into the c -number functions so that their compositions could be obtained by the star product that is noncommutative but associative. Therefore, the observables would be classical functions of the phase space. Quantum structure is constructed by replacing pointwise products of classical observables of the phase space (x, Π) , by star product [16]. The product of two smooth functions, say $f = f(x, \Pi)$ and $g = g(x, \Pi)$, on a Poisson-Moyal manifold is given by

$$f * g = \sum_{n=0}^{\infty} (i\hbar)^n C_n(f, g), \quad (1)$$

where \hbar plays the role of the deformation parameter. The first term $C_0(f, g) = fg$ denotes the common pointwise product of f and g . Also, the coefficients $C_n(f, g)$ are bidifferential operators, where their product is noncommutative [17]. These coefficients satisfy the following properties

$$\begin{aligned} C_0(f, g) &= fg, \\ C_1(f, g) - C_1(g, f) &= \{f, g\}, \\ \sum_{i+j=n} C_i(C_j(f, g), h) &= \sum_{i+j=n} C_i(f, C_j(g, h)), \end{aligned} \quad (2)$$

where $\{f, g\}$ denotes the ordinary Poisson bracket. In equations (2), the first expression means that in the limit, $\hbar \rightarrow 0$, the star product of f and g agrees with the pointwise products of these two functions in classical phase space. The second expression shows that at the lowest order of the deformation parameter, the $*$ -commutator $[f, g]_* = f * g - g * f$ tends to the Poisson bracket. In flat spaces, there is a special star product which has long been known. In this case, the components of the Poisson tensor can be considered constant and consequently it is possible to define the following Moyal star product [14]

$$\begin{aligned} f(x, \Pi) *_M g(x, \Pi) &= f(x, \Pi) \exp\left(\frac{i\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_\Pi - \overleftarrow{\partial}_\Pi \overrightarrow{\partial}_x)\right) g(x, \Pi) \\ &= f\left(x + \frac{i\hbar}{2} \overrightarrow{\partial}_\Pi, \Pi - \frac{i\hbar}{2} \overrightarrow{\partial}_x\right) g(x, \Pi). \end{aligned} \quad (3)$$

The last equality in (3) suggests that for a Moyal star product $A *_M B$ of two functions A and B the Weyl-Wigner correspondence reads

$$(A *_M B)(x, \Pi) = A\left(x + \frac{i\hbar}{2} \partial_\Pi, \Pi - \frac{i\hbar}{2} \partial_x\right) B(x, \Pi), \quad (4)$$

where now $A\left(x + \frac{i\hbar}{2} \partial_\Pi, \Pi - \frac{i\hbar}{2} \partial_x\right)$ should be understood as an operator acting on $\mathcal{C}^\infty(\text{phase space})$. According to this prescription we have to replace the position and momentum variables in A by pseudo-operators containing position and momentum and their derivatives, that is

$$x \longrightarrow x + \frac{i\hbar}{2} \partial_\Pi, \quad \Pi \longrightarrow \Pi - \frac{i\hbar}{2} \partial_x, \quad (5)$$

instead of the usual correspondence $x \longrightarrow x$, $\Pi \longrightarrow -\frac{i\hbar}{2} \partial_x$. These pseudo-operators sometimes carry the name Bopp pseudo-operators. These pseudo-operators act, not on functions defined on \mathbb{R}^D as ordinary Weyl operators do, but on

functions (or distributions) defined on the noncommutative phase space $\mathbb{R}^D \oplus \mathbb{R}^D$. In fact, Bopp pseudo-differential operators is a tool of choice for the study of deformation quantization which it reduces to a Weyl calculus of a particular type. Also, equation (4) shows that noncommutative quantum mechanics can also be reduced to Bopp calculus from an operator point of view. One of the most important components of deformation quantization is the Wigner quasiprobability distribution function [18]. In fact, it is a generating function for all spatial autocorrelation functions of a given quantum mechanical wave function [19]. The relation of Wigner function with wave function of system $\psi_n(x)$ in a $2D$ -dimensional phase space is

$$W_n(x, \Pi) = \frac{1}{(2\pi\hbar)^D} \int \psi_n^*(x - \frac{\hbar}{2}y) \psi_n(x + \frac{\hbar}{2}y) e^{-i\frac{\Pi \cdot y}{\hbar}} d^D y. \quad (6)$$

In this formalism of quantum mechanics, expectations of observables and transition amplitudes are phase space integrals of c -number functions, weighted by the Wigner function, as in statistical mechanics.

In the next section we will investigate the quantum cosmology of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, filled with radiation plus dust or cosmic string. Using Wigner function we will show that the deformed cosmology predicts a good correlation with the corresponding classical cosmology. Natural units $c = \hbar = 1$ are used throughout the paper.

2 Deformation quantization of FLRW cosmology

There exist two fundamentally different approaches to quantization of general relativity: The particle physics programme and the canonical quantum gravity programme. In the first programme, the basic entity is the graviton, the quantum of the gravitational field. Such a particle is deemed to propagate in a background Minkowski spacetime, η , and like all elementary particles, is associated with a specific representation of the Poincaré group. By fixing the background topology and differential structure of spacetime manifold to be that of Minkowski spacetime, and then splinting spacetime into background, $\eta_{\mu\nu}$, and dynamical, $h_{\mu\nu}$, parts, the field $h_{\mu\nu}$ is quantized by the standard field methods under the assumption that the gravitational interaction, like other standard matter interactions, involves the exchange of gravitons. On the other hand, the canonical approach to quantum gravity starts with a reference foliation of spacetime with respect to which the appropriate canonical variables are defined. In this sense, quantization of gravitation is quantization of the metrical structure of spacetime, which satisfies a dynamical principle and dynamical equations. The essence of this approach is expressed by the name quantum geometrodynamics. To study the nature of the difficulties of quantum geometrodynamics, it is convenient to use a particular minisuperspace approximation. The model nature of these studies is a consequence of various factors, of which the main are: 1) a minisuperspace of definite symmetry is selected and it is assumed that the symmetry is preserved in the quantization process. 2) to maintain a definite symmetry of minisuperspace, one requires a material source (perfect fluid, Yang-Mills fields, scalar fields or spinor fields), which is also quantized when gravitation is quantized. It is clear that in the domain of quantum geometrodynamics (Planck scale) the description of the material source will differ from the one adopted in quantum field theory, and therefore in quantum geometrodynamics the source is taken into account only formally by the addition of new degrees of freedom to the equations that describe the dynamical geometry [20]. Due to the quantum nature of the model, as a first approximation, the matter content should be described by some sorts of fields, as done in [21]. However, general exact solutions are hard to find at the presence of Yang-Mills, scalar or spinor fields and the Hilbert space structure is obscure and it is a subtle matter to recover the notion of a semiclassical time [21, 22]. In addition, since the Bekenstein Bound tells us that the information content of the very early Universe is zero, the only physical variable we have to take into account is the scale factor of the Universe, and the density and pressure of the matter field. Hence, we only have to quantize the FLRW universe for a perfect fluid [23]. Also, perfect fluid has the advantage of introducing a variable which can naturally be identified with time, leading to a well-defined Hilbert space structure [24]. Another attractive feature of the phenomenological perfect fluid description of matter degree of freedom is that it allows us to treat the barotropic equation of state which allows us to obtain general exact solutions. The line element of spatially flat FLRW universe is

$$ds^2 = -N^2(t)dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \quad (7)$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor. The action functional that consists of a gravitational part and a matter part when the matter field is considered as a perfect fluid is given by [25]

$$S = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x - \int \rho \sqrt{-g} d^4x, \quad (8)$$

where g is the determinant of spacetime metric and R is the Ricci scalar and $\rho = \sum_i \rho_i$ is the total energy density. We assume a universe filled with noninteracting perfect fluids with energy densities of $\rho_i = \rho_{0i} (\frac{a}{a_0})^{-3(\omega_i+1)}$ where ω_i

denotes the equation of state parameter of i -th component of fluid and ρ_{i0} is the energy density at the measuring epoch. The action (8) reduce to

$$S = -\frac{3V_3}{8\pi G} \int Na^3 \left(\frac{a}{N^2} \left(\frac{da}{dt} \right)^2 + \frac{8\pi G}{3} \sum_i \rho_{0i} \left(\frac{a}{a_0} \right)^{-3(\omega_i+1)} \right) dt, \quad (9)$$

where $V_3 = \int d^3x$ is the spacial volume of 3-metric. Let us rewrite the action (9) in terms of measurable quantities in cosmology. This will help us to compare the quantum cosmological model with the corresponding classical model. First, we define a new lapse function by $\tilde{N} = \frac{N}{x}$. By writing the energy density of various components of fluid in terms of corresponding density parameters, $\Omega_i = \frac{8\pi G \rho_{0i}}{3H_0^2}$ (H_0 is the Hubble parameter at the measuring epoch) the energy densities will be $\rho_i = \frac{3H_0^2 \Omega_i}{8\pi G} \left(\frac{a}{a_0} \right)^{-3(1+\omega_i)}$. Also if we use a new dimensionless scale factor defined by $x = \frac{a}{a_0}$ and a new dimensionless time coordinate by $\eta = H_0 t$, the Lagrangian of model in conformal frame up to a multiplicative constant $\frac{3V_3 a_0^3 H_0}{4\pi G}$, will be

$$\mathcal{L} = -\frac{1}{2} \left(\frac{\dot{x}^2}{\tilde{N}} + \tilde{N} \sum_i \Omega_i x^{1-3\omega_i} \right), \quad (10)$$

where over dot denotes differentiation respect to η . The conjugate momentum to the scale factor, x , and the primary constraint are given by

$$\Pi_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{\dot{x}}{\tilde{N}}, \quad \Pi_{\tilde{N}} = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{N}}} = 0. \quad (11)$$

Consequently, the Hamiltonian corresponding to Lagrangian (10) will be

$$H = \tilde{N} \left(-\frac{\Pi_x^2}{2} + \frac{1}{2} \sum_i \Omega_i x^{1-3\omega_i} \right). \quad (12)$$

In Hamiltonian (12), \tilde{N} is a Lagrange multiplier; therefore, it enforces the Hamiltonian constraint

$$\mathcal{H} = -\frac{\Pi_x^2}{2} + \frac{1}{2} \sum_i \Omega_i x^{1-3\omega_i} = 0, \quad (13)$$

where \mathcal{H} denotes the super-Hamiltonian. The super-Hamiltonian at the initial time $\eta_0 = t_0 H_0$ reduces to well-known relation between density parameters $\sum_i \Omega_i = 1$.

The deformation quantization of this simple model is accomplished straightforwardly by replacing the ordinary products of the observables in phase space by the Moyal product. Therefore, Hamiltonian constraint (13) becomes the Moyal-Wheeler-DeWitt (MWDW) equation by replacing the classical Hamiltonian (13) with its deformed counterpart [13]

$$\mathcal{H}(x, \Pi_x) *_M W_n(x, \Pi_x) = 0. \quad (14)$$

Since the $*_M$ -product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments (see, Eq.(4)). Therefore, the MWDW equation (14) is equivalent to

$$\mathcal{H} \left(x + \frac{i}{2} \vec{\partial}_{\Pi_x}, \Pi_x - \frac{i}{2} \vec{\partial}_x \right) W(x, \Pi_x) = 0. \quad (15)$$

Let us now first investigate the classical-quantum correlation in a radiation-dust filled universe. In this case, the classical super-Hamiltonian (13) will be

$$\Pi_x^2 - \Omega_d x - \Omega_r = 0, \quad (16)$$

where Ω_d and Ω_r denote the density parameters of dust and radiation respectively, obeying relation $\Omega_d + \Omega_r = 1$ at the measuring epoch, η_0 . Also, the MWDW equation (15) will be

$$\left(\left(\Pi_x - \frac{i}{2} \vec{\partial}_x \right)^2 - \Omega_d \left(x + \frac{i}{2} \vec{\partial}_{\Pi_x} \right) - \Omega_r \right) W = 0. \quad (17)$$

Note that we should order the kinetic term as $\Pi_x^2 \rightarrow x^{-\alpha} * \Pi_x * x^\alpha * \Pi_x = \Pi_x^2 - i\alpha x^{-1} * \Pi_x$, where α takes into account the factor ordering ambiguity. This is equivalent to the factor ordering in corresponding WDW equation given by $\Pi_x^2 \rightarrow -x^{-\alpha} \partial_x (x^\alpha \partial_x)$ [26]. In this paper we will consider the choice $\alpha = 0$ ordering which is equivalent to the Laplace-Beltrami operator of conformal frame in the corresponding WDW equation. The Wigner function is real, hence by separation the real and imaginary parts of MWDW equation (17) we obtain two coupled partial differential equations

$$\begin{aligned} \left(-\frac{1}{4}\partial_x^2 - \Omega_d x + \Pi_x^2 - \Omega_r\right) W(x, \Pi_x) &= 0, \\ (\Pi_x \partial_x + \frac{\Omega_d}{2} \partial_{\Pi_x}) W(x, \Pi_x) &= 0. \end{aligned} \quad (18)$$

The first equation does not involve the partial derivatives of Π_x . However, the second phase space equation enforces a special symmetry on the solutions. The solution of second partial differential equation of (18) is, $W = f(\Pi_x^2 - \Omega_d x)$, where f denotes a general real function. With the help of definition of new variable $\zeta = \Pi_x^2 - \Omega_d x$ and the relation $\frac{\partial^2 W(x, \Pi_x)}{\partial x^2} = \Omega_d^2 \frac{d^2 f(\zeta)}{d\zeta^2}$ following from the chain rule, the first partial differential equation of (18) reduces to following second order ordinary differential equation

$$-\frac{\Omega_d^2}{4} \frac{d^2 f(\zeta)}{d\zeta^2} + (\zeta - \Omega_r) f(\zeta) = 0. \quad (19)$$

The finite value solution of this equation is

$$W(x, \Pi_x) = \mathcal{N} Ai \left(\left(\frac{2}{\Omega_d} \right)^{\frac{2}{3}} (\Pi_x^2 - \Omega_d x - \Omega_r) \right), \quad (20)$$

where $\mathcal{N} = \frac{1}{2\pi} \left(\frac{2}{\Omega_d} \right)^{\frac{2}{3}}$ and $Ai(\xi)$ denotes the Airy function of first kind. The locus of extremums of the above Wigner function is the following deformed super-Hamiltonian

$$\Pi_x^2 - \Omega_d x - \Omega_r + \left(\frac{\Omega_d}{2} \right)^{\frac{2}{3}} a_n = 0, \quad (21)$$

where a_n are the zeroes of derivative Airy functions $\frac{d}{d\xi} Ai(-\xi)|_{\xi=a_n} = 0$. See Table (1) for the first several terms of a_n sequences. Hence, equation (21) presents the most probable cosmological solutions. These solutions (for various values

Table 1: Negatives of zeroes of Ai' for $n = 1, 2, 3$.

| n | a_n |
|---|------------|
| 1 | 1.01879... |
| 2 | 3.24819... |
| 3 | 4.82009... |

of a_n) are the same as the original classical solution (16) but with modified value of the density parameter of radiation given by

$$\tilde{\Omega}_r(n) = \Omega_r - \left(\frac{\Omega_d}{2} \right)^{\frac{2}{3}} a_n. \quad (22)$$

It is obvious that all of this solutions (that we have for various values of a_n) are non-singular if

$$\Omega_r \left(\frac{\Omega_d}{2} \right)^{\frac{2}{3}} < a_1. \quad (23)$$

Fig.(1) shows the Wigner function of model with corresponding classical trajectory in phase space. It is seen that a good correlation exists between the quantum quasiprobability distribution shown in this figure and the classical trajectory in phase space for large values of scale factor, x , where the universe is dust dominated. The observable difference of classical and quantum cosmology is in the values of density parameter of radiation in super-Hamiltonians (16) and (21). At the very early times the deformed universe is radiation dominated and singularity free. At the late times the predictions of both theories are the same and there is an exact correlation between classical and quantum universes.

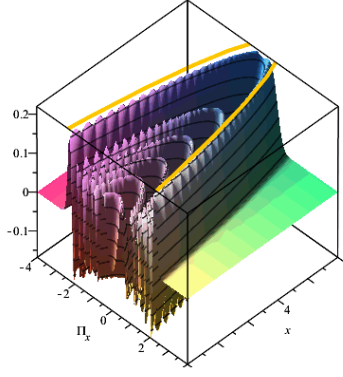


Figure 1: The Wigner function of dust and radiation filled universe. The corresponding classical trajectory is denoted by goldline loci. This figure is plotted for $\Omega_d = \Omega_r = 0.5$.

Let us now investigate the classical-quantum correlation of a universe filled with cosmic string perfect fluid, with $\omega_{cs} = -\frac{1}{3}$. In this case, the classical super-Hamiltonian (13) will be

$$\Pi_x^2 - \Omega_{cs}x^2 = 0. \quad (24)$$

Also, the MWDW equation (15) given by

$$\left((\Pi_x - \frac{i}{2}\vec{\partial}_x)^2 - \Omega_{cs}(x + \frac{i}{2}\vec{\partial}_{\Pi_x})^2 \right) W = 0. \quad (25)$$

Separation of real and imaginary parts of the above equation gives two independent equations

$$(\Omega_{cs}x\partial_{\Pi_x} + \Pi_x\partial_x) W = 0, \quad (26)$$

$$(\Pi_x^2 - \Omega_{cs}x^2 - \frac{1}{4}\partial_x^2 + \frac{\Omega_{cs}}{4}\partial_{\Pi_x}^2) W = 0.$$

The finite value solution at the classical singularity is given by

$$W(x, \Pi_x) = \frac{1}{2\pi\sqrt{\Omega_{cs}}} J_0 \left(\frac{\Pi_x^2 - \Omega_{cs}x^2}{\sqrt{\Omega_{cs}}} \right), \quad (27)$$

where $J_0(\xi)$ is the Bessel function of order zero. In this very simple model, the locus of extremums of Wigner function are given by

$$\Pi_x^2 - \Omega_{cs}x^2 - \sqrt{\Omega_{cs}}j_n = 0, \quad (28)$$

where j_n are the zeroes of derivative Bessel function. Table (2) shows the first several zeroes of $\frac{dJ_0(\xi)}{d\xi}$. The modified

Table 2: Zeroes of Bessel function, j_n , for $n = 1, 2, 3$.

| n | j_n |
|---|-----------|
| 1 | 0 |
| 2 | 3.8317... |
| 3 | 7.0155... |

super-Hamiltonian (28) represents a universe filled with cosmic string and radiation fluids, where the density parameter of radiation is given by

$$\Omega_r = \sqrt{\Omega_{cs}j_n}. \quad (29)$$

Note that the radiation part of (28) has totally quantum origin and the corresponding classical universe is filled only with the cosmic string fluid. In the late times, where the universe is cosmic string dominated, the predictions of both models are the same. But for very small values of scale factor, the emerged universes for various values of j_n are non-singular and radiation dominated. The first zero of derivative Bessel function is zero, consequently for $j_0 = 0$ the prediction of quantum cosmology is a cosmic string filled singular universe, same as the corresponding classical universe (24). Fig.(2) shows the corresponding Wigner function with corresponding classical trajectory in phase space. For large values of scale factor, x , there is a good correlation between the quantum quasiprobability distribution and the classical trajectory in phase space.

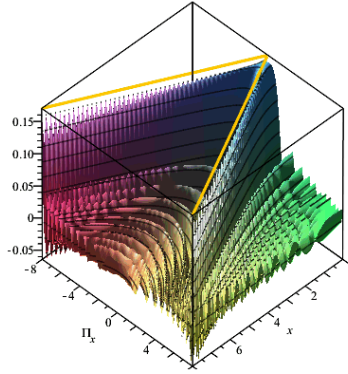


Figure 2: The Wigner function of cosmic string filled universe. The corresponding classical trajectory is denoted by goldline loci. This figure is plotted for $\Omega_{cs} = 1$.

3 Conclusion

In this paper we studied the deformation or phase space quantization of a flat quantum FLRW model, whose matter is either a fluids of radiation plus dust, or cosmic string. We show that the peaks of Wigner quasiprobability distribution function of quantum universe filled with cosmic string are coincide with the emerged universes whose are filled with cosmic string and radiation fluids. Also, for a universe filled by the radiation and dust fluids, the peaks of Wigner quasiprobability distribution function are coincide with the same classical universes but with modified density parameter of radiation. Consequently the behaviour of emerged quantum universes are different with the corresponding classical models for the very small values of scale factors, where the quantum universes are non-singular. On the other hand, for large values of scale factor, the emerged quantum universes are coincide with corresponding classical universes. This behavior can be interpreted that the classical approximation of the Universe becomes better and better as the universe expands. We believe the above results offer an insight into the relation between classical and quantum cosmologies and that the particular simple models that we have studied may serve as useful starting point for more ambitious investigation. We are aware that our results are obtained within a very simple as well as restricted setting. Nevertheless, we think they are intriguing and provide motivation for subsequent research works. A wider analysis, with less restrictive cosmologies and/or other matter fields, should follow. The presence of phenomenological fluid matter was broadly used in, e.g., [28] so that exact solutions of the (simplified) WDW equation could be obtained. Using instead, e.g., a scalar, spinor or Yang-Mills fields would be more generic and more realistic from the point of view of matter interaction with the gravitational field in a high energy regime, where quantum effects can be expected. We are leaving the above enticing research lines for future works.

References

- [1] J.B. Hartle, S.W. Hawking and T. Hertog, Phys. Rev. D **77**, 123537 (2008), [arXiv:0803.1663].
- [2] T. Dereli, M. Önder and Robin W. Tucker, Class. Quantum Grav. **10**, 1425 (1993).
- [3] F.G. Alvarenga, J.C. Fabris, N.A. Lemos and G.A. Monerat, Gen. Rel. Grav. **34**, 651 (2002), [arXiv:gr-qc/0106051]; N.A. Lemos, J. Math. Phys. **37**, 1449 (1996); G.A. Monerat, G. Oliveira-Neto, E.V. Corrêa Silva, L.G. Ferreira Filho, P. Romildo Jr., J.C. Fabris, R. Fracalossi, F.G. Alvarenga and S.V.B. Goncalves, Phys. Rev. D **76**, 024017 (2007); F.G. Alvarenga, A.B. Batista and J.C. Fabris, Int. J. Mod. Phys. D **14**, 291 (2005), [arXiv:gr-qc/0404034]; P. Pedram and S. Jalalzadeh, Phys. Lett. B **659**, 6 (2008), [arXiv:0711.1996]; P. Pedram, S. Jalalzadeh and S.S. Gousheh, Phys. Lett. B **655**, 91 (2007), [arXiv:0708.4143]; P. Pedram and S. Jalalzadeh, Phys. Rev. D **77**, 123529 (2008), [arXiv:0805.4099].
- [4] R. Omnès, *The Interpretation of Quantum Mechanics*, Princeton University Press, Princeton, (1994).
- [5] H.D. Zeh, Found. Phys. **1**, 69 (1970); E. Joos and H.D. Zeh, Z. Phys. B **59**, 223 (1985); W.H. Zurek, Phys. Today **44**, 36 (1991).
- [6] V.F. Mukhanov, in *Physical Origins of Time Asymmetry*, ed. by J.J. Halliwell, J. Pérez-Mercader and W.H. Zurek, Cambridge University Press (1994); H.D. Zurek, in *Decoherence and the Appearance of Classical World in Quantum Theory* Springer-Verlag, Berlin, (1996).

- [7] N. Pinto-Neto, Brazilian J. Phys. **30**, 330 (2000).
- [8] M. Gell-Mann and J.B. Hartle, in *Complexity, Entropy and the Physics of Information*, ed. by W.H. Zurek, Addison-Wesley, (1990).
- [9] J.P. Paz and W.H. Zurek, Phys. Rev. D **48**, 2728 (1993).
- [10] N. Pinto-Neto and J.C. Fabris, Class. Quantum Grav. **30**, 143001 (2013), [arXiv:1306.0820].
- [11] F. Shojai and A. Shirinifard, Int. J. Mod. Phys. D **14**, 1333 (2005), [arXiv:gr-qc/0504138]; P. Pedram and S. Jalalzadeh, Phys. Lett. B **660**, 1 (2008), [arXiv:0712.2593]; P. Peter and N. Pinto-Neto, Phys. Rev. D **78**, 063506 (2008), [arXiv:0809.2022].
- [12] M. Fathi, S. Jalalzadeh and P.V. Moniz, Eur. Phys. J. C **76**, 527 (2016), [arXiv:1609.04488].
- [13] F. Antonsen, [arXiv:gr-qc/9712012]; Phys. Rev. D **56**, 920 (1997); H. Quevedo and J.G. Tafoya, Gen. Relativ. Gravit. **37**, 2083 (2005) [arXiv:gr-qc/0401088]; H. García-Compeán, F.J. Turrubiates, Int. J. Mod. Phys. A **26**, 5241 (2011); R. Cordero, H. García-Compeán and F.J. Turrubiates Phys. Rev. D **83**, 125030 (2011), [arXiv:1102.4379]; M. Rashki and S. Jalalzadeh, Phys. Rev. D **91**, 023501 (2015), [arXiv:1412.3950].
- [14] J. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
- [15] H. Weyl, Z. Phys. **46**, 1 (1927); E. Wigner, Phys. Rev. **40**, 749 (1932); H. J. Groenewold, Physica (Utrecht) **12**, 405 (1946); F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Ann. Phys. (N.Y.) **111**, 111 (1978).
- [16] M. Kontsevich, Lett. Math. Phys. **48**, 35 (1999), [arXiv:math/9904055].
- [17] A.C. Hirschfeld and P. Henselder, Am. J. Phys. **70**, 537 (2002), [arXiv:quant-ph/0208163].
- [18] M. Hillery, R. F. Oconnell, M. C. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984); T. Curtright, D. Fairlie, and C. Zachos, Phys. Rev. D **58**, 025002 (1998); M. Levanda and V. Fleurov, Ann. Phys. (N.Y.) **292**, 199 (2001); C. Zachos, Int. J. Mod. Phys. A **17**, 297 (2002).
- [19] B. V. Fedosov, J. Differential Geom. **40**, 213 (1994); M. Gadella, Fortschr. Phys. **43**, 229 (1995).
- [20] V.G. Lapchinskii and V.A. Rubakov, Theor. Math. Phys. **33**, 1076 (1977).
- [21] C. Kiefer, Phys. Rev. D **38**, 1761 (1988).
- [22] C.J. Isham, [arXiv:gr-qc/9210011].
- [23] F.J. Tipler, Rept. Prog. Phys. **68**, 897 (2005), [arXiv:0704.3276].
- [24] M.J. Gotay and J. Demaret, Phys. Rev. D **28**, 2402 (1983); F.G. Alvarenga, J.C. Fabris, N.A. Lemos and G.A. Monerat, Gen. Rel. Grav. **34**, 651 (2002), [arXiv:gr-qc/0106051]; P. Pedram, M. Mirzaei, S. Jalalzadeh and S. S. Gousheh, Gen. Rel. Grav. **40**, 1663 (2008), [arXiv:0711.3833]; B. Vakili, Class. Quantum. Grav. **27**, 025008 (2010), [arXiv:0908.0998]; P. Pedram, S. Jalalzadeh and S.S. Gousheh, Phys. Lett. B. **655**, 91 (2007), [arXiv:0708.4143]; P. Pedram, S. Jalalzadeh and S.S. Gousheh, Class. Quantum. Grav. **24**, 5515 (2007), [arXiv:0709.1620].
- [25] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, United Kingdom, 1973).
- [26] R. Steigl and F. Hinterleitner, Class. Quantum Grav. **23**, 3879 (2006), [arXiv:gr-qc/0511149].
- [27] M. Hillery et al., Phys. Rep. **106**, 121 (1984).
- [28] J.H. Kung, Gen. Relat. Gravit. **27**, 35 (1995), [arXiv:hep-th/9302016]; J. Demaret, Gen. Relat. Gravit. **11**, 453 (1979); T. Banks, W. Fischler and L. Mannelli, Phys. Rev. D **71**, 123514 (2005) [arXiv:hep-th/0408076]; B. Majumder, Int. J. Mod. Phys. D **22**, 1350079 (2013), [arXiv:1307.5273]; D. Atkatz and H. Pagels, Phys. Rev. D **25**, 2065 (1982); P.S. Letelier and J.P.M. Pitelli, Phys. Rev. D **82**, 104046 (2010), [arXiv:1010.3054]; B. Majumder and N. Banerjee, Gen. Relativ. Gravit. **45**, 1 (2013), [arXiv:1208.4481]; S. Nojiri and S.D. Odintsov, Phys. Lett. B **562**, 147 (2003), [arXiv:hep-th/030311].